

# Cryptanalysis of Stream Cipher Alpha1

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**Abstract.** Komninos, Honary and Darnell recently proposed stream cipher Alpha1. Alpha1 is based on A5/1 and is claimed to be much safer than A5/1. However Alpha1 is insecure: a 29-bit feedback shift register could be recovered with about 3000 known plaintext bits with  $O(2^{29})$  operations. The rest of the message could be recovered in the broadcast application in which the same message is encrypted with different keys.

## 1 Introduction

The stream cipher Alpha1 [6] was proposed recently to strengthen A5/1. A5/1 is used to protect the mobile communication privacy and is with extremely simple structure: three LFSRs (64 bits in total) are used and the output is obtained by XORing the most significant bits of those registers. The LFSRs go/stop according to the majority rule: each register provides one bit (clocking tap) and the majority bit is computed. The registers whose clocking taps agreeing with the majority bit are clocked.

There have been several attacks on A5/1 [1,5,3,4]. The real time known-plaintext cryptanalysis of A5/1 were given by Biryukov, Shamir and Wagner [4] by exploiting two major weaknesses of A5/1: the relatively small number of internal states ( $2^{64}$ ) and the poor choice of the clocking taps. As a cipher with small number of internal states, A5/1 is vulnerable to the general time-memory tradeoff attack on stream ciphers (which is discovered independently by Golic [5] and Babbage [2]). The poor choice of clocking taps enables special states being accessed easily and the efficiency of the time-memory tradeoff attack can be improved significantly.

To strengthen A5/1, Komninos, Honary and Darnell proposed Alpha1. Alpha1 is with four LFSRs (128 bits in total). Three registers go/stop according to two groups of clocking taps while one register is regularly clocked. The attacks against A5/1 seem no longer applicable to Alpha1. Unfortunately, an additional binary AND operation is used in Alpha1 in order to achieve high linear complexity. This AND operation causes a disaster to Alpha1: the combination of the output of those irregularly clocked LFSRs is not uniformly distributed. The statistical information could be applied to recover the register which is regularly clocked. The design of Alpha1 reminds us that it is dangerous to modify the internal structure of a cipher.

After recovering the regularly clocked register, information of the message is leaked. If a message is encrypted with a number of different keys and those ciphertext are known (in some broadcast applications), the rest of the message could be recovered. If a message is longer than  $2^{29} - 1$  bits and is encrypted with different keys, the ciphertext only attack is applicable to recover the message.

This paper is organized as follows. Section 2 introduces the Alpha1 stream cipher. The attack against Alpha1 is given in Section 3. The message recovery in the broadcast application is studied in Section 4. Section 5 concludes this paper.

## 2 Stream Cipher Alpha1

Alpha1 is built from four short linear feedback shift registers (LFSRs) of lengths 29, 31, 33 and 35 bits, which are denoted as  $R_1, R_2, R_3$  and  $R_4$  respectively. Four primitive polynomials  $p_1(x), p_2(x), p_3(x)$  and  $p_4(x)$  are used for  $R_1, R_2, R_3$  and  $R_4$  respectively. They are given in Table 1.

**Table 1.** Primitive Polynomials

$p_1(x)$	$x^{29} + x^{27} + x^{24} + x^8 + 1$
$p_2(x)$	$x^{31} + x^{28} + x^{23} + x^{18} + 1$
$p_3(x)$	$x^{33} + x^{28} + x^{24} + x^4 + 1$
$p_4(x)$	$x^{35} + x^{30} + x^{22} + x^{11} + x^6 + 1$

We denote  $R_1, R_2, R_3$  and  $R_4$  as

$$R_1 = \sum_{i=0}^{28} R_{1,i} \cdot x^i, \quad R_2 = \sum_{i=0}^{30} R_{2,i} \cdot x^i$$

$$R_3 = \sum_{i=0}^{32} R_{3,i} \cdot x^i, \quad R_4 = \sum_{i=0}^{34} R_{4,i} \cdot x^i$$

The output  $y$  of Alpha1 is given as

$$y = R_{1,28} \oplus R_{2,30} \oplus R_{3,32} \oplus R_{4,34} \oplus (R_{2,30} \& R_{3,32})$$

where ‘&’ is the binary AND operator.

$R_1$  is shifted every clock cycle.  $R_2, R_3$  and  $R_4$  are clocked in a stop/go fashion according to the following majority rule:  $R_2$  is clocked if  $R_{2,10}$  agrees with the majority bit of  $(R_{2,10}, R_{3,22}, R_{4,11})$  and  $R_{2,21}$  agrees with the majority bit of  $(R_{2,21}, R_{3,10}, R_{4,24})$ ;  $R_3$  is clocked if  $R_{3,22}$  agrees with the majority bit of  $(R_{2,10}, R_{3,22}, R_{4,11})$  and  $R_{3,10}$  agrees with the majority bit of  $(R_{2,21}, R_{3,10}, R_{4,24})$ ;  $R_4$  is clocked if  $R_{4,11}$  agrees with the majority bit of  $(R_{2,10}, R_{3,22}, R_{4,11})$  and  $R_{4,24}$  agrees with the majority bit of  $(R_{2,21}, R_{3,10}, R_{4,24})$ . We list in Table 2 all the 16 cases that  $R_2, R_3$  and  $R_4$  stop/go. The designers of Alpha1 listed only 13 cases and estimated wrongly that each  $R_2, R_3$  and  $R_4$  move with probability  $\frac{7}{13}$  [6]. We need to mention here that those 16 cases do not occur independently. We will discuss the probabilities of those cases in Section 3.

**Table 2.** Registers ( $R_2, R_3, R_4$ ) Being Shifted At One Clock Cycle

	Condition 1	Condition 2	Registers Being Shifted
Case 1	$R_{2,10} = R_{3,22} \neq R_{4,11}$	$R_{2,21} = R_{3,10} \neq R_{4,24}$	$R_2, R_3$
Case 2	$R_{2,10} = R_{3,22} \neq R_{4,11}$	$R_{2,21} \neq R_{3,10} = R_{4,24}$	$R_3$
Case 3	$R_{2,10} = R_{3,22} \neq R_{4,11}$	$R_{2,21} = R_{4,24} \neq R_{3,10}$	$R_2$
Case 4	$R_{2,10} = R_{3,22} \neq R_{4,11}$	$R_{2,21} = R_{3,10} = R_{4,24}$	$R_2, R_3$
Case 5	$R_{2,10} \neq R_{3,22} = R_{4,11}$	$R_{2,21} = R_{3,10} \neq R_{4,24}$	$R_3$
Case 6	$R_{2,10} \neq R_{3,22} = R_{4,11}$	$R_{2,21} \neq R_{3,10} = R_{4,24}$	$R_3, R_4$
Case 7	$R_{2,10} \neq R_{3,22} = R_{4,11}$	$R_{2,21} = R_{4,24} \neq R_{3,10}$	$R_4$
Case 8	$R_{2,10} \neq R_{3,22} = R_{4,11}$	$R_{2,21} = R_{3,10} = R_{4,24}$	$R_3, R_4$
Case 9	$R_{2,10} = R_{4,11} \neq R_{3,22}$	$R_{2,21} = R_{3,10} \neq R_{4,24}$	$R_2$
Case 10	$R_{2,10} = R_{4,11} \neq R_{3,22}$	$R_{2,21} \neq R_{3,10} = R_{4,24}$	$R_4$
Case 11	$R_{2,10} = R_{4,11} \neq R_{3,22}$	$R_{2,21} = R_{4,24} \neq R_{3,10}$	$R_2, R_4$
Case 12	$R_{2,10} = R_{4,11} \neq R_{3,22}$	$R_{2,21} = R_{3,10} = R_{4,24}$	$R_2, R_4$
Case 13	$R_{2,10} = R_{3,22} = R_{4,11}$	$R_{2,21} = R_{3,10} \neq R_{4,24}$	$R_2, R_3$
Case 14	$R_{2,10} = R_{3,22} = R_{4,11}$	$R_{2,21} \neq R_{3,10} = R_{4,24}$	$R_3, R_4$
Case 15	$R_{2,10} = R_{3,22} = R_{4,11}$	$R_{2,21} = R_{4,24} \neq R_{3,10}$	$R_2, R_4$
Case 16	$R_{2,10} = R_{3,22} = R_{4,11}$	$R_{2,21} = R_{3,10} = R_{4,24}$	$R_2, R_3, R_4$

### 3 Cryptanalysis of Alpha1 – Recover $R_1$

In this section, we show that  $R_1$  could be determined with about 3000 known plaintext bits by exploiting the non-uniform distribution of the output of  $R_2$ ,  $R_3$  and  $R_4$ . We write the Alpha1 output  $y$  alternatively as

$$y = R_{1,28} \oplus (R_{2,30} | R_{3,32}) \oplus R_{4,34}$$

where ‘|’ is the binary inclusive OR operator. Let

$$y' = (R_{2,30} | R_{3,32}) \oplus R_{4,34},$$

then

$$y = R_{1,28} \oplus y'.$$

Denote the output of Alpha1 as  $Y$  and let the sequences consisting of  $y'$  and  $R_{1,28}$  be  $Y'$  and  $Y''$  respectively. Obviously  $Y = Y' \oplus Y''$ . Since  $R_1$  is clocked regularly, there are only  $2^{29}$  possible values for  $Y''$ . If the bits of  $Y'$  are not uniformly distributed, those statistical properties could be used to recover  $Y''$  ( $= R_1$ ). We will show below that 00, 01, 10 and 11 in  $Y'$  are not uniformly distributed.

We start with computing the probabilities of those 16 cases. By analyzing Table 2, we know that: if Case 2 or 5 appears, only Case 1, 2, 5 and 6 may appear in the next clock cycle and each one appears with equal probability  $\frac{1}{4}$ ; if Case 3 or 9 appears, only Case 1, 3, 9 and 11 may appear in the next clock cycle and

each one appears with equal probability  $\frac{1}{4}$ ; if Case 7 or 10 appears, only Case 6, 7, 10 and 11 may appear in the next clock cycle and each one appears with equal probability  $\frac{1}{4}$ ; if Case 1, 4, 6, 8, 11, 12, 13, 14, 15 or 16 appears, in the next clock cycle each of those 16 cases appears with equal probability  $\frac{1}{16}$ . These 16 cases form a Markoff Chain. We compute the steady state probabilities and list them in Table 3.

**Table 3.** The Probability of Each Case

	Probability		Probability
Case 1	$\frac{3}{28}$	Case 9	$\frac{2}{28}$
Case 2	$\frac{2}{28}$	Case 10	$\frac{2}{28}$
Case 3	$\frac{2}{28}$	Case 11	$\frac{3}{28}$
Case 4	$\frac{1}{28}$	Case 12	$\frac{1}{28}$
Case 5	$\frac{2}{28}$	Case 13	$\frac{1}{28}$
Case 6	$\frac{3}{28}$	Case 14	$\frac{1}{28}$
Case 7	$\frac{2}{28}$	Case 15	$\frac{1}{28}$
Case 8	$\frac{1}{28}$	Case 16	$\frac{1}{28}$

At the  $i$ th clock cycle, if Case 1, 4 or 13 appears, only  $R_2$  and  $R_3$  are clocked,  $(y'_i, y'_{i-1})$  would be 00, 01, 10, 11 with probability  $\frac{5}{16}, \frac{3}{16}, \frac{3}{16}, \frac{5}{16}$  respectively; if Case 2, 3, 5 or 9 appears, only one of  $R_2$  and  $R_3$  is clocked,  $(y'_i, y'_{i-1})$  would be 00, 01, 10, 11 with probability  $\frac{3}{8}, \frac{1}{8}, \frac{1}{8}, \frac{3}{8}$  respectively; if Case 6, 7, 8, 10, 11, 12, 14, 15 or 16 appears,  $R_4$  is clocked and  $R_2$  or  $R_3$  may be clocked,  $(y'_i, y'_{i-1})$  would be 00, 01, 10, 11 with equal probability  $\frac{1}{4}$ . Thus  $(y'_i, y'_{i-1})$  would be 00, 01, 10, 11 with probability  $\frac{19}{64}, \frac{13}{64}, \frac{13}{64}, \frac{19}{64}$  respectively.

In the following we compute how much information of  $R_1$  could be recovered if  $(n + 1)$  bits of the key stream  $Y$  are known. Denote this  $(n + 1)$ -bit key stream as  $Y_n$ . Randomly set the initial value of  $R_1$  as a 29-bit non-zero  $z$  and generate an  $(n + 1)$ -bit output  $Y''_{n,z}$  which is related to  $Y_n$ . Let

$$Y'_{n,z} = Y_n \oplus Y''_{n,z}$$

Let  $n_{00}, n_{01}, n_{10}$  and  $n_{11}$  represent the number of 00, 01, 10 and 11 in  $Y'_{n,z}$  respectively. If  $z$  is the correct  $R_1$ , 00, 01, 10 and 11 would appear in  $Y'_{n,z}$  with probability  $\frac{19}{64}, \frac{13}{64}, \frac{13}{64}, \frac{19}{64}$  respectively. Let

$$G_{n,x} = \{(n_{00}, n_{01}, n_{10}, n_{11}) \mid n_{00}, n_{11} \geq n \cdot (\frac{19}{64} - x), n_{01}, n_{10} \leq n \cdot (\frac{13}{64} + x) \text{ and } n_{00} + n_{01} + n_{10} + n_{11} = n\}$$

$x$  is chosen such that if  $z$  is the correct  $R_1$ ,  $(n_{00}, n_{01}, n_{10}, n_{11}) \in G_{n,x}$  with probability  $P_1$  close to 1. For the wrong  $z$ ,  $(n_{00}, n_{01}, n_{10}, n_{11}) \in G_{n,x}$  with probability  $P_2$ . These two probabilities are given as:

$$P_1 = \sum_{(n_{00}, n_{01}, n_{10}, n_{11}) \in G_{n,x}} C(n; n_{00}, n_{01}, n_{10}, n_{11}) \left(\frac{19}{64}\right)^{n_{00}+n_{11}} \left(\frac{13}{64}\right)^{n_{01}+n_{10}} \quad (1)$$

$$P_2 = \sum_{(n_{00}, n_{01}, n_{10}, n_{11}) \in G_{n,x}} C(n; n_{00}, n_{01}, n_{10}, n_{11}) \left(\frac{1}{4}\right)^n \quad (2)$$

where  $C(n; n_{00}, n_{01}, n_{10}, n_{11})$  is the multinomial coefficient  $n!/(n_{00}!n_{01}!n_{10}!n_{11}!)$ . The information leakage  $\Delta s$  of  $R_1$  is given as

$$\Delta s = 29 - \left( - \sum_{i=1}^{|A|} \frac{P_1}{|A|} \cdot \log_2 \frac{P_1}{|A|} - \sum_{i=1}^{|B|} \frac{1 - P_1}{|B|} \cdot \log_2 \frac{1 - P_1}{|B|} \right) \quad (3)$$

where

$$\begin{aligned} A &= \{z \mid \text{For } Y'_{n,z}, (n_{00}, n_{01}, n_{10}, n_{11}) \in G_{n,x}\} \\ B &= \{z \mid \text{For } Y'_{n,z}, (n_{00}, n_{01}, n_{10}, n_{11}) \notin G_{n,x}\} \end{aligned}$$

Let  $|A| \approx 2^{29} \cdot P_2 + 1$  and  $|B| = 2^{29} - |A|$ , (3) is approximated as

$$\Delta s \approx P_1 \cdot \log_2 \frac{P_1}{P_2 + 2^{-29}} + (1 - P_1) \cdot \log_2 \frac{1 - P_1}{1 - P_2 - 2^{-29}} \quad (4)$$

It is difficult to compute  $P_1$  and  $P_2$  directly for  $n$  with large value. We approximate (1) and (2) with the multivariate normal distribution:

$$P_1 = \int_{n \cdot (\frac{19}{64} - x) - \frac{1}{2}}^n \int_{n \cdot (\frac{19}{64} - x) - \frac{1}{2}}^n \int_0^{n \cdot (\frac{13}{64} + x) + \frac{1}{2}} f(z_1, z_2, z_3) dz_1 dz_2 dz_3 \quad (5)$$

where

$$\begin{aligned} f(z_1, z_2, z_3) &= \frac{1}{\sqrt{(2\pi)^3 (\det \mathbf{V})}} e^{-(z-\mathbf{u})^T \mathbf{V}^{-1} (z-\mathbf{u})/2} \\ \mathbf{z} &= \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} \frac{13}{64} n \\ \frac{19}{64} n \\ \frac{19}{64} n \end{pmatrix} \\ \mathbf{V} &= \begin{pmatrix} \frac{13}{64} \times \frac{51}{64} n & -\frac{13}{64} \times \frac{19}{64} n & -\frac{13}{64} \times \frac{19}{64} n \\ -\frac{64}{19} \times \frac{64}{13} n & \frac{64}{19} \times \frac{64}{45} n & -\frac{64}{19} \times \frac{64}{19} n \\ -\frac{19}{64} \times \frac{64}{13} n & -\frac{19}{64} \times \frac{19}{64} n & \frac{19}{64} \times \frac{45}{64} n \end{pmatrix} \end{aligned}$$

$P_2$  is approximated in a similar way as  $P_1$  except that

$$\mathbf{u} = \begin{pmatrix} \frac{n}{4} \\ \frac{n}{4} \\ \frac{n}{4} \end{pmatrix} \quad \mathbf{V} = \begin{pmatrix} \frac{3}{16} n & -\frac{1}{16} n & -\frac{1}{16} n \\ -\frac{1}{16} n & \frac{3}{16} n & -\frac{1}{16} n \\ -\frac{1}{16} n & -\frac{1}{16} n & \frac{3}{16} n \end{pmatrix} \quad (6)$$

From (4), (5) and (6), we compute the information leakage  $\Delta s$  of  $R_1$  for different values of  $n$ . The numerical integration function in MATHEMATICA is used in the computation. Different values of  $x$  are tested for each  $n$  so that the maximum value of  $\Delta s$  could be obtained. The results are given in Table 4.

**Table 4.**  $\Delta s_{\max}$  vs  $n$

$n$	$x$	$P_1$	$P_2$	$\Delta s_{\max}$
512	$\frac{10}{512}$	0.645	$2^{-9.953}$	5.477
768	$\frac{14}{768}$	0.706	$2^{-13.41}$	8.588
1024	$\frac{17}{1024}$	0.728	$2^{-17.47}$	11.88
1280	$\frac{20}{1280}$	0.752	$2^{-21.44}$	15.32
1536	$\frac{24}{1536}$	0.796	$2^{-24.58}$	18.79
1792	$\frac{29}{1792}$	0.847	$2^{-26.92}$	21.94
2048	$\frac{35}{2048}$	0.897	$2^{-28.53}$	24.39
2304	$\frac{42}{2304}$	0.937	$2^{-29.45}$	26.09
2560	$\frac{49}{2560}$	0.962	$2^{-30.42}$	27.21
2816	$\frac{56}{2816}$	0.977	$2^{-31.44}$	27.93
3072	$\frac{64}{3072}$	0.987	$2^{-31.84}$	28.35

From Table 4 about 28 bits of  $R_1$  could be recovered with about 3000 known plaintext bits. We implemented the attack for  $n = 3072$ ,  $x = \frac{64}{3072}$ . The attack is repeated for 100 times for different initial values of  $R_1$ ,  $R_2$ ,  $R_3$  and  $R_4$ . Each experiment takes about 3.6 hours on Pentium IV 1.7GHz. These 100 experiments give 90 correct  $R_1$  and 11 wrong  $R_1$  (80 experiments give only the correct  $R_1$ ; 11 experiments give one correct  $R_1$  and one wrong  $R_1$ ; 9 experiments give no value for  $R_1$ ). The experiment results are close to the estimated results given in Table 4.

### 4 Message Recovery in the Broadcast Applications

In Section 3, we recovered  $R_1$  with known plaintext attack. With the knowledge of  $R_1$  each bit of the ciphertext leaks  $2^{-5.3}$ -bit information since 00, 01, 10 and 11 appear in the output of  $R_2$ ,  $R_3$  and  $R_4$  with probabilities  $\frac{19}{64}$ ,  $\frac{13}{64}$ ,  $\frac{13}{64}$  and  $\frac{19}{64}$  respectively. In some broadcast applications, one message is encrypted with different keys and sent to different users. For 31, 63, 127 and 255 different keys, the information leakage for each plaintext bit is 0.405, 0.650, 0.880 and 0.986 bit respectively.

If the message is longer than  $(2^{29} - 1)$  bits, even the ciphertext only attack could be applied to recover a lot of information if the message is encrypted with sufficiently large number of different keys. Assume that the message is  $(2^{30} - 2)$  bits long, denote the plaintext and ciphertext as  $m_1 \parallel m_2$  and  $c_1 \parallel c_2$  respectively, where  $m_1$ ,  $m_2$ ,  $c_1$  and  $c_2$  are  $(2^{29} - 1)$ -bit long and  $\parallel$  denotes concatenation. The output of  $R_2$ ,  $R_3$  and  $R_4$  is denoted as  $Y'_1 \parallel Y'_2$ . Since the output of  $R_1$  is with period  $2^{29} - 1$ , we obtain  $c_1 \oplus c_2 = (Y'_1 \oplus Y'_2) \oplus (m_1 \oplus m_2)$ . The 00, 01, 10 and 11 appear in  $Y'_1 \oplus Y'_2$  with probabilities  $\frac{265}{1024}$ ,  $\frac{247}{1024}$ ,  $\frac{247}{1024}$  and  $\frac{265}{1024}$  respectively. The information leakage is  $2^{-10.1}$  for each bit of  $m_1 \oplus m_2$ . For 255, 511, 1023 and 2047 different keys, the information leakage for each bit of  $m_1 \oplus m_2$  is 0.135, 0.252, 0.442 and 0.690 bit respectively.

## 5 Conclusions

The attack given in this paper recovers the 29-bit register  $R_1$  from around 3000-bit known plaintext. The rest of the message could be recovered in the broadcast applications. It is an open problem whether  $R_2$ ,  $R_3$  and  $R_4$  could be recovered by analyzing their non-uniformly distributed output.

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